On the equilibrium state of two rotating charged masses in General Relativity

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Abstract

This paper is a direct continuation of our publication [3] where it was found the exact solution of the Einstein-Maxwell equations for two static sources of Reissner-Nordstrom type in the state of the physical equilibrium. Here we present the exact solution of these equations for the case of two rotating charged sources and we proved the existence of the physical equilibrium state also for this general case.

1 Introduction

In the non-relativistic physics two particles can be in equilibrium if the product of their masses is equal to the product of their charges (we use the units $G = c = 1$). However, the question on the existence of an analogue of such equilibrium state in General Relativity is far to be trivial. Besides the natural mathematical complications, in General Relativity arise two different types of the "point" centers, namely Black Hole (BH) and Naked Singularity (NS) and one need to consider all three configurations BH - BH, NS - NS and BH - NS separately. Yet in each case the notion of a physically sensible distance between these objects should be defined.

When the Inverse Scattering Method (ISM) have been adopted for integration of the Einstein and Einstein-Maxwell equations it was shown on the exact mathematical level that Black Holes and Naked Singularities represent nothing else but stationary axially symmetric solitons. Then by the ISM machinery one can obtain the infinite families of exact stationary axially symmetric solutions of these equations containing such solitons centralized at different points of the symmetry axis. The formal construction of such solutions do not represents any difficulties apart of the routine calculations in the framework of the well developed procedure how to insert a number of solitons into a given background

spacetime. However, it is quite intricate task to single out from these families the physically reasonable constructions which correspond to a real equilibrium states of charged Black Holes and Naked Singularities interacting with each other. The point is that in general the stationary axially symmetric solitonic solutions possess some features which are unacceptable from the physical point of view. These unwanted traits are due the presence in the solutions exotic peculiarities of the following four types: (i) NUT parameters, (ii) angle deficit at the points of the symmetry axis, (iii) closed time-like curves around that parts of symmetry axis which are out of the sources and (iv) magnetic charges. The NUT parameters are incompatible with asymptotic flatness of the spacetime at spatial infinity. The angle deficit is the well known conical singularity violating the local Euclidness of space at the points of symmetry axis (it can be treated as some singular external strut or string preventing the sources to fall onto or to run away each other). Keeping in mind the physical applications we also should avoid of any excess of closed timelike curves with respect to those already existing inside the sources as an inseparable part of their inner structure. Also magnetic charges should be excluded since their presence contradicts the Maxwell theory. All four aforementioned phenomena have nothing to do with a real equilibrium of the physical bodies and the corresponding equilibrium solution should be free of such pathologies. To deliberate solution from them one need to place the set of the free parameters of the solution under some additional restrictions which can be written in the form of some system of algebraic equations. The problem is that these equations, even for the simplest case of two objects, are extremely complicated and it is difficult to resolve them in an exact analytical form in order to see directly whether they have physically appropriate solutions compatible with the existence of a positive definite distance between the sources.

However, the aforementioned nuisances constitute the real troubles only in the general case of the rotating sources. The static case is much more simple and it would be not an overstatement to say that for the case of 2 non-rotating charged objects the problem have been solved completely. The first indications that two static charged masses can stay in real physical equilibrium without any struts between them and without any other pathologies came from the results of Bonnor [1] and Perry and Cooperstock [2]. In [1] it was analyzed the equilibrium condition for a charged test particle in the Reissner-Nordstrom field and it was shown that such test body can be at rest in the field of the Reissner-Nordstrom source only if they both are either extreme (the charge equal to mass) or one of them is of BH type (the charge is less than the mass) and the other is of NS type (the charge is grater than the mass). There is no way for equilibrium in cases when both masses are either of NS type or both are of BH type. The more solid arguments in favour of existence of a static equilibrium configuration for the Black Hole - Naked Singularity system was presented in [2], where both sources have been treated exactly, that is no one of the components was considered as test particle. These results have been obtained with the aid of numerical calculations and three examples of numerical solutions of the equilibrium equation have been demonstrated. These solutions can correspond

to the equilibrium configurations free of struts, though the authors have not been able to show the existence of a positive definite distance between the sources. The authors of [2] also reported that a number of numerical experiments for two Black Holes and for two Naked Singularities showed the negative outcomes, i.e. all tested sets of the parameters was not in power to satisfy the equilibrium equation. These findings were in agreement with Bonnor's test particle analysis.

The explicit analytical resolution of the problem in static case have been presented in paper [3] where it was constructed the exact analytic solution of the Einstein-Maxwell equations for two sources separated by the well defined positive distance and free of struts or of any other unphysical properties [1]. We showed also that such solution indeed exists only for the BH-NS system and it is impossible to have the similar equilibrium state for the pair BH-BH or NS-NS. After these results the natural question arises whether the analogous equilibrium exists for two rotating sources. It turn out that the answer is affirmative and in the present paper we demonstrate the exact equilibrium solution of the Einstein-Maxwell equation for two rotating charged objects one of which is a Black Hole and another represents a Naked Singularity.

2 On the general properties of solitonic solutions

Metric and electromagnetic potentials for any stationary axisymmetric case in cylindrical Weyl coordinates (t, ρ, z, φ) take the forms:

$$ds^2 = -f\left(d\rho^2 + dz^2\right) + g_{tt}dt^2 + 2g_{t\varphi}dtd\varphi + g_{\varphi\varphi}d\varphi^2, \tag{1}$$

$$g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2. \tag{2}$$

$$A_t = A_t(\rho, z), \quad A_\varphi = A_{\varphi t}(\rho, z), \quad A_\rho = 0, \quad A_z = 0\ , \tag{3}$$

where all metric coefficients depend only on the variables ρ, z and the signature of the metric corresponds to the following signs of the metric coefficients in the diagonal case: $f > 0$, $g_{\varphi\varphi} < 0$, $g_{tt} > 0$. The gravitational solitons (in the context of ISM) as exact solutions of pure gravity Einstein equations have been introduced in the papers [5, 6]. The generalization of this technique for the coupled gravitational and electromagnetic fields was constructed in [7]. Its more detailed description can be found in [8] and in the book [9]. In this generalized approach one starts from some given background solution of the Einstein-Maxwell equations and stick into it any desired number of solitons. We have to do here with the linear spectral differential equations (Lax pair) for the 3×3 matrix function $\Psi(\rho, z, w)$, where w is a complex spectral parameter independent of coordinates ρ, z. First of all we choose some background solution of the Einstein-Maxwell equations and find from the Lax pair the corresponding background spectral matrix $\Psi_0(\rho, z, w)$.Using the ISM dressing procedure it is

[1]The more details of the derivation of this solution an interested reader can find in paper [4].

possible to find explicitly the spectral matrix $\Psi_n(\rho, z, w)$, corresponding to the new solution containing n solitons inserted to the background spacetime and one can extract the new metric and new electromagnetic potentials from this Ψ_n. The solitonic field added to the background can be characterized by the matrix $\Psi_n \Psi_0^{-1} - I$ which is a meromorphic matrix function tending to zero in the limit $w \to \infty$ and having n simple poles in the complex plane of the parameter w (one pole for each soliton).

In pure gravity case some of these poles can be located at the real axis of the w-plane and the corresponding sources have horizons (that is they are of the BH type) while complex poles generate objects with naked singularities. However, in the presence of electromagnetic field the formal machinery of the ISM developed in [7] in general does not permit for poles to be located at real axis which means that by this method one can produce solutions containing sources only of the NS type.[2] Nevertheless, also in this case after one obtains the final form of solution it is possible to forget the way how it was derived and to continue the solution analytically in the space of its parameters in order to get the complete family containing solutions with real metric of the physical signature and with horizons as well. However, the technical procedure how to do this is simple only for the case of one-solitonic solution (that is for the Kerr-Newman case) and some simple enough generalization of such procedure was found also for two static solitonic objects [10]. In the general case of two rotating sources (the corresponding 12-parametric solitonic solution have been constructed in ([11])) this task is much more complicated. Fortunately, there is an effective way to get over this difficulty. Because we need to construct solution of the BH-NS type we can consider the Kerr-Newman black hole as new background (instead of the flat spacetime) and insert to it one soliton of NS type. This is exactly what can be done easily with the generating technique proposed in [7] and what we are interested in. The exact expressions (in terms of the Ernst potentials) for the solution together with the proof that all conditions of the physical equilibrium can be satisfied are given below.

3 The BH-NS system in a stationary state

Our solution depends on twelve independent real constant parameters

$$\{m_0, a_0, b_0, q_0, \mu_0\}, \quad \{m_s, a_s, b_s, q_s, \mu_s\}, \quad l = z_2 - z_1 > 0 \quad \text{and} \quad c_0 \ , \qquad (4)$$

where the parameters with the suffix "0" are related to the background solution (black hole) and the parameters with the suffix "s" are the parameters of a soliton; all parameters are real. The parameter l (which was chosen positive for definiteness) characterizes a coordinate distance between the sources because z_1 and z_2 determine respectively the location of a black hole and a naked singularity on the axis. The constant c_0 is an arbitrary multiplier in front of the

[2] We said "in general" because it can be shown that the ISM considered in [7] can be adjust also to the real w-poles but only for that special restriction on the parameters of the solutions which correspond to the extreme Black Holes.

metric coefficient f in (1) which should be chosen in accordance, e.g., with the condition of regularity of the axis at spatial infinity. It is convenient to use two functions of the enumerated parameters – a real σ_0 and an pure imaginary σ_s determined by the relations

$$\sigma_0^2 = m_0^2 + b_0^2 - a_0^2 - q_0^2 - \mu_0^2 \geq 0, \qquad \sigma_s^2 = m_s^2 + b_s^2 - a_s^2 - q_s^2 - \mu_s^2 \leq 0. \quad (5)$$

Our solution is stationary and axisymmetric and depends on two Weyl coordinates ρ, z. However, it is more convenient to express it in terms of so called "bipolar" coordinates – two pairs of polar coordinate centered respectively at the location of a black hole (the coordinates with the suffix 1) and at the location of a naked singularity (the coordinates with the suffix 2). Of course, these four coordinates should satisfy two additional constraints and each of these four coordinates can be expressed in terms of Weyl coordinates ρ, z. The corresponding defining relations take the forms

$$\rho = \sqrt{x_1^2 - \sigma_0^2}\sqrt{1 - y_1^2} = \sqrt{x_2^2 - \sigma_s^2}\sqrt{1 - y_2^2} \quad \text{and} \quad z = z_1 + x_1 y_1 = l + z_1 + x_2 y_2, \quad (6)$$

though, it is worth to note that z_1 is not an essential parameter because it determines a shift of the whole configuration of the fields and their sources along the axis. The inverse relations of bipolar coordinates (x_1, y_1) corresponding to real σ_0 in terms of Weyl coordinates are

$$\begin{aligned} x_1 &= \frac{1}{2}\left[\sqrt{(z - z_1 + \sigma_0)^2 + \rho^2} + \sqrt{(z - z_1 - \sigma_0)^2 + \rho^2}\right], \\ y_1 &= \frac{1}{2\sigma_0}\left[\sqrt{(z - z_1 + \sigma_0)^2 + \rho^2} - \sqrt{(z - z_1 - \sigma_0)^2 + \rho^2}\right]. \end{aligned} \quad (7)$$

For the coordinates (x_2, y_2) corresponding to imaginary σ_s ($\sigma_s^2 < 0$) the similar relations are more complicate ($z_2 = l + z_1$):

$$\begin{aligned} x_2 &= \sqrt{\frac{1}{2}\left[(z - z_2)^2 + \rho^2 + \sigma_s^2\right] + \sqrt{\frac{1}{4}\left[(z - z_2)^2 + \rho^2 + \sigma_s^2\right]^2 - \sigma_s^2 (z - z_2)^2}}, \\ y_2 &= \frac{z - z_2}{x_2}. \end{aligned} \quad (8)$$

Sometimes it is convenient also to use instead of pairs of coordinates (x_1, y_1) and (x_2, y_2) the pairs of quasi-spherical coordinates (r_1, θ_1) and (r_2, θ_2):

$$\begin{aligned} x_1 &= r_1 - m_0\,, & x_2 &= r_2 - m_s\,, \\ y_1 &= \cos\theta_1\,. & y_2 &= \cos\theta_2\,. \end{aligned} \quad (9)$$

The Ernst potentials and metric coefficient f of this solution are:

$$\mathcal{E} = 1 - \frac{2(m_0 - ib_0)}{R_1} - \frac{2(m_s - ib_s)}{R_2}, \qquad \Phi = \frac{q_0 + i\mu_0}{R_1} + \frac{q_s + i\mu_s}{R_2}, \quad (10)$$

$$\frac{1}{R_1}=\frac{x_2+ia_sy_2+K_1(x_2-\sigma_sy_2)+L_1(x_1+\sigma_0y_1)+S_0\,(x_2+\sigma_sy_2)}{D}, \qquad (11)$$
$$\frac{1}{R_2}=\frac{x_1+ia_0y_1+K_2(x_1-\sigma_0y_1)+L_2(x_2+\sigma_sy_2)}{D},$$

$$\begin{aligned}D=&(x_1+ia_0y_1+m_0-ib_0)\,[x_2+ia_sy_2+m_s-ib_s+S_0\,(x_2+\sigma_sy_2)]-\\&-[m_0-ib_0-K_2(x_1-\sigma_0y_1)-L_2(x_2+\sigma_sy_2)]\times\\&\times[m_s-ib_s-K_1(x_2-\sigma_sy_2)-L_1(x_1+\sigma_0y_1]\,,\end{aligned} \qquad (12)$$

$$K_1=\frac{ia_s-\sigma_s}{\sigma_0+\sigma_s+l},\;L_1=\frac{(m_0+ib_0)(m_s-ib_s)-(q_0-i\mu_0)\,(q_s+i\mu_s)}{(ia_0-\sigma_0)(\sigma_1+\sigma_2+l)}, \qquad (13)$$
$$K_2=\frac{ia_0-\sigma_0}{\sigma_0+\sigma_s-l},\;L_2=\frac{(m_0-ib_0)(m_s+ib_s)-(q_0+i\mu_0)\,(q_s-i\mu_s)}{(ia_s-\sigma_s)(\sigma_0+\sigma_s-l)},$$

$$S_0=-\frac{\sigma_0Y_s\bar{Y}_s}{\sigma_s\,(\sigma_0^2+a_0^2)\,(ia_s-\sigma_s)(\sigma_0+\sigma_s-l)}, \qquad (14)$$

$$Y_s=(m_0-ib_0)\,(q_s+i\mu_s)-(m_s-ib_s)\,(q_0+i\mu_0)\,, \qquad (15)$$

$$f=c_0\frac{D\bar{D}}{(x_1^2-\sigma_0^2y_1^2)\,(x_2^2-\sigma_s^2y_2^2)}. \qquad (16)$$

Here and in what follows the bar over letters means complex conjugation.

4 Asymptotics at spatial infinity

It is easy to see that at spatial infinity, i.e. for $\rho^2+z^2\to\infty$, the Ernst potentials (10)-(14) have the following asymptotical behaviour:

$$\mathcal{E}=1-\frac{2(M-iB)}{r}+O(\frac{1}{r^2}),\qquad \Phi=\frac{Q_e+iQ_m}{r}+O(\frac{1}{r^2}),\qquad r=\sqrt{\rho^2+z^2}\,, \qquad (17)$$

where M and B are the total gravitational mass and total NUT parameter of the configuration and Q_e and Q_m are its total electric and magnetic charges. Simple calculations show that these parameters are:

$$M=\mathrm{Re}\left[\frac{(m_0-ib_0)(1+K_1+L_1+S_0)+(m_s-ib_s)(1+K_2+L_2)}{1+S_0-(K_1+L_1)(K_2+L_2)}\right], \qquad (18)$$

$$B=-\mathrm{Im}\left[\frac{(m_0-ib_0)(1+K_1+L_1+S_0)+(m_s-ib_s)(1+K_2+L_2)}{1+S_0-(K_1+L_1)(K_2+L_2)}\right] \qquad (19)$$

$$Q_e=\mathrm{Re}\left[\frac{(q_0+i\mu_0)\,(1+K_1+L_1+S_0)+(q_s+i\mu_s)\,(1+K_2+L_2)}{1+S_0-(K_1+L_1)(K_2+L_2)}\right] \qquad (20)$$

$$Q_m=\mathrm{Im}\left[\frac{(q_0+i\mu_0)\,(1+K_1+L_1+S_0)+(q_s+i\mu_s)\,(1+K_2+L_2)}{1+S_0-(K_1+L_1)(K_2+L_2)}\right] \qquad (21)$$

As for the metric coefficient f it should be subject to the constraint $f \to 1$ in the limit $r \to \infty$. This condition can be satisfied by an appropriate choice of the coefficient c_0. It is easy to see that for this we have to choose

$$c_0 = |1 + S_0 - (K_1 + L_1)(K_2 + L_2)|^{-2} \ . \tag{22}$$

Then if we would like to have a configuration without NUT parameter and magnetic charge we should impose on the parameters of the solution the restrictions $B = 0$ and $Q_m = 0$.

5 On the closed time-like curves

If at some points of the axis (where $\rho = 0$) one have $g_{t\varphi} \neq 0$, this implies (in accordance with the relation between the metric coefficients in Weyl coordinates $g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2$) that near these points $g_{\varphi\varphi} > 0$. Such inequality means that near these points of the axis the coordinate lines of the periodic (azimuth angle) coordinate, being closed lines, are time-like. To avoid such trouble it is necessary to demand that on every part of the axis $g_{t\varphi}$ should vanish. As it follows directly from the Einstein - Maxwell equations, on the axis of symmetry $\rho = 0$ the value $\Omega = g_{t\varphi}/g_{tt}$ is independent of z and therefore, it is a constant. However, this constant can be different on different disconnected parts of the axis separated by the sources. Therefore, to exclude the existence of closed time-like curves near the axis, first of all we should impose two conditions

$$\Omega_- = \Omega_i = \Omega_+ \ , \tag{23}$$

where Ω_-, Ω_i and Ω_+ are the constants which are the values of $g_{t\varphi}/g_{tt}$ on the negative, intermediate and positive parts of the axis respectively. If the conditions (23) are satisfied, the corresponding common value of Ω can be reduced to zero by a "global" coordinate transformation of the form $t' = t + a\varphi$, and $\varphi' = \varphi$ with an appropriate constant a.[3]

In order to satisfy the conditions (23) we need to calculate constants $\Omega_+ - \Omega_-$ and $\Omega_i - \Omega_-$ and put both of them to zero. Calculations show that the first constant take simple form:

$$\Omega_+ - \Omega_- = -4B \ , \tag{24}$$

where B is the total NUT parameter given by the formula (19), while the second parameter is much more complicated:

$$\Omega_i - \Omega_- \equiv -4B - \frac{\omega_\times \overline{\omega}_\times}{(a_\times + i\sigma_s)} + \frac{(1+2\delta)}{(1-2\delta)} \frac{\mathcal{H}_0 \overline{\mathcal{H}}_0}{(a_\times + i\sigma_s)\mathcal{W}_o}. \tag{25}$$

[3]It is worth to note here that this transformation actually is not an admissible coordinate transformation (because φ is a periodic coordinate but t is not) and its correct interpretation is connected with some global restruction of the space-time manifold (some cut-and-past procedure) such that the coordinate lines of φ become non-closed while the other closed lines appear and they become the coordinate lines of a new asimutal coordinate φ'.

The explicit expression for δ takes the form:

$$\delta = \frac{\sigma_0(m_0 m_s + b_0 b_s - q_0 q_s - \mu_0 \mu_s)}{\sigma_0(l^2 - \sigma_0^2 - \sigma_s^2 - 2a_0 a_s) + (l\sigma_0 + \sigma_0^2 - i a_0 \sigma_s)(l - \sigma_0 - \sigma_s) S_0} \ , \qquad (26)$$

where S_0 follows from (14) and

$$\omega_\times = m_\times - i b_\times + i(a_\times + i\sigma_s), \qquad \mathcal{W}_o = (l^2 - \sigma_0^2 - \sigma_s^2)^2 - 4\sigma_0^2\sigma_s^2 \ , \qquad (27)$$

$$\begin{array}{l} \mathcal{H}_0 = -2i(l - \sigma_s - i a_0 + m_0 + i b_0)\overline{X}_\times + \\ (a_\times + i\sigma_s - i m_\times + b_\times)[(l - \sigma_s)^2 - \sigma_0^2] + \\ +2(a_\times + i\sigma_s)\left[(m_0 + i b_0)(l - \sigma_s + i a_0) + \sigma_0^2 + a_0^2\right] \end{array} \qquad (28)$$

$$X_\times = (m_0 + i b_0)(m_\times - i b_\times) - (q_0 - i\mu_0)\,(q_\times + i\mu_\times)\,. \qquad (29)$$

In these formulas we used the new parameters denoted by the same letters but with subscript "$\times$". These new constants are defined by the relations:

$$m_\times = M - m_0 \ , \ b_\times = B - b_0 \ , \ q_\times = Q_e - q_0 \ , \ \mu_\times = Q_m - \mu_0 \ , \qquad (30)$$

$$a_\times + i\sigma_s = \frac{\Gamma\bar{\Gamma}\sqrt{c_0}}{(a_s + i\sigma_s)\sqrt{\mathcal{W}_o}} \ , \qquad (31)$$

$$\Gamma = (m_0 + i b_0)\,(m_s - i b_s) - (q_0 - i\mu_0)\,(q_s + i\mu_s) - (a_s + i\sigma_s)\,(a_0 - i\sigma_s - i l) \ , \qquad (32)$$

where parameters $c_{0,} M, B, Q_e, Q_m$ have been defined earlier by the relations (18)-(22).

6 On the conical singularity at the axis

If conditions (23) are satisfied, this does not mean yet that the geometry on each part of the axis is regular since at the points of different parts of the axis the local Euclidness of spatial geometry still may occur to be violated. This behaviour looks like on the surface of a cone near its vortex, where the ratio of the length of a circle (surrounding the vortex) to its "radius" is not equal to 2π. In the solution, on any surface $z = const$ intersecting the axis, the length and radius of the circles $\rho = const$ are represented asymptotically for $\rho \to 0$ by the expressions $L = 2\pi\sqrt{-g_{\varphi\varphi}}$ and $R = \sqrt{f}\rho$ respectively. Using the mentioned above relation for metric coefficients in Weyl coordinates $g_{tt}g_{\varphi\varphi} - g_{t\varphi}^2 = -\rho^2$ and the condition $g_{t\varphi} \to 0$ discussed just above, we obtain that the condition of the local Euclidness of the geometry at these points, i.e. the condition $L/(2\pi R) \to 1$ for $\rho \to 0$, is equivalent to the following constraints:

$$P_- = P_i = P_+ = 1, \qquad P \equiv f g_{tt} \ , \qquad (33)$$

where P_-, P_i and P_+ are the values of the product $f g_{tt}$ respectively on the negative, intermediate and positive parts of the axis. (In accordance with the Einstein - Maxwell equations, the product $f g_{tt}$ is constant on a part of the axis

where $g_{t\varphi} = 0$, however, these constants again may occur to be different for different disconnected parts of the axis.) To obtain conditions which should be imposed on the parameters of our solution providing the equations (33) to be satisfied, we have to analyze the behaviour of metric components on different parts of the axis of symmetry. Let's do this.

Negative semi-infinite part of the axis: $\{\rho = 0, -\infty < z < z_1 - \sigma_0\}$. On this part of the axis

$$x_1 = z_1 - z, \quad x_2 = l + z_1 - z, \quad y_1 = y_2 = -1, \tag{34}$$

and the direct calculations of the metric component g_{tt} and of the factor f lead to the following expressions

$$g_{tt} = \frac{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]}{c_0 D_- \overline{D}_-}, \tag{35}$$
$$f = \frac{c_0 D_- \overline{D}_-}{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]} ,$$

where c_0 has been defined in (22) and D_- denotes the value of D (defined by (12)) on the negative semi-infinite part of the axis. From these expressions we see that the condition (33) is satisfied automatically on the negative semi-infinite part of the axis.

Positive semi-infinite part of the axis: $\{\rho = 0, z_1 + l < z < \infty\}$. On this part of the axis

$$x_1 = z - z_1, \quad x_2 = z - z_1 - l, \quad y_1 = y_2 = 1, \tag{36}$$

and here for the metric component g_{tt} and the conformal factor we have the similar expressions

$$g_{tt} = \frac{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]}{c_0 D_+ \overline{D}_+}, \tag{37}$$
$$f = \frac{c_0 D_+ \overline{D}_+}{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]} ,$$

where D_+ denotes the value of D on the positive semi-infinite part of the axis. From these expressions we see that the condition (33) is satisfied identically also on the positive semi-infinite part of the axis.

Intermediate part of the axis: $\{\rho = 0, z_1 + \sigma_0 < z < z_1 + l\}$. On this part of the axis

$$x_1 = z - z_1, \quad x_2 = l + z_1 - z, \quad y_1 = 1, \quad y_2 = -1, \tag{38}$$

and the corresponding expressions for g_{tt} and the coefficient f on the axis take the forms:

$$g_{tt} = \frac{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]}{c_0 D_i \overline{D}_i} \left(\frac{1-2\delta}{1+2\delta}\right)^2, \tag{39}$$
$$f = \frac{c_0 D_i \overline{D}_i}{[(z-z_1)^2 - \sigma_0^2][(z-z_1-l)^2 - \sigma_s^2]} ,$$

where parameter function δ is the same which have been defined already by the formula (26). As follows from these last expression the condition (33) on the intermediate part of the axis (i.e. the equation $P = 1$) is equivalent to the constraint $\delta = 0$.

7 Physical magnetic and electric charges

To obtain a physical result we have to exclude from the solution magnetic charges of both sources. To calculate the physical values of magnetic charges (note that formal parameters μ_0 and μ_s are not physical magnetic charges, they coincide with them only in the limit of infinite distance between the sources $l \to \infty$) we should consider the magnetic fluxes coming from closed space-like surfaces surrounding each charged center and apply the Gauss theorem. In this way we can find the physical magnetic charges $\mu_0^{(ph)}, \mu_s^{(ph)}$ (as well as physical electric charges $q_0^{(ph)}, q_s^{(ph)}$) of each source calculating the corresponding Komar-like integrals. The detailed procedure how to do this have been described in the section "Physical parameter of the sources" in paper [4]. The results of these calculations are:

$$q_0^{(ph)} = q_0 + \mathrm{Re}\, F \ , \ \mu_0^{(ph)} = \mu_0 + \mathrm{Im}\, F \ , \tag{40}$$

$$q_s^{(ph)} = q_\times - \mathrm{Re}\, F \ , \ \mu_s^{(ph)} = \mu_\times - \mathrm{Im}\, F \ , \tag{41}$$

where

$$F = (q_\times + i\mu_\times)\,\frac{(a_\times + i\sigma_s - im_\times + b_\times)}{2(a_\times + i\sigma_s)} - \frac{\mathcal{L}_0\mathcal{H}_0}{2\mathcal{W}_o(a_\times + i\sigma_s)}\frac{(1+2\delta)}{(1-2\delta)} \ , \tag{42}$$

and we introduced here the new parameter polynomial:

$$\begin{aligned} \mathcal{L}_0 = \left[(l+\sigma_s)^2 - \sigma_0^2\right](q_\times + i\mu_\times) + \\ +2\left[X_\times - i(a_\times + i\sigma_s)(l + \sigma_s - ia_0)\right](q_0 + i\mu_0)\,. \end{aligned} \tag{43}$$

The physically acceptable solution corresponds to the restrictions $\mu_0^{(ph)} = \mu_s^{(ph)} = 0$.

8 Summary for the equilibrium conditions

Here we present a summary of the conditions which provide an equilibrium of two interacting Kerr-Newman sources and which should be satisfied by the parameters of the whole configuration. Thus, if we fix an arbitrary constant multiplier in the metric coefficient f as it was described above, we have an eleven-parameter solution. The first of the equilibrium conditions is the vanishing of a NUT parameter B. Then from (19) we have:

$$\mathrm{Im}\left[\frac{(m_0 - ib_0)(1 + K_1 + L_1 + S_0) + (m_s - ib_s)(1 + K_2 + L_2)}{1 + S_0 - (K_1 + L_1)(K_2 + L_2)}\right] = 0 \tag{44}$$

The second condition provides the local Euclidness of the geometry on the axis (i.e. the absence of the conical points) that is equivalent to the vanishing of the parameter δ which was defined by the expression (26):

$$m_0 m_s + b_0 b_s = q_0 q_s + \mu_0 \mu_s \tag{45}$$

The third condition follows from the absence of the closed time-like curves. This demand consists of two restrictions $\Omega_+ - \Omega_- = 0$ and $\Omega_i - \Omega_- = 0$. Taking into account equations (44) and (45), these two restrictions gives now only one independent relation $\Omega_i - \Omega_- = 0$ which acquire the form (see (24) and (25)):

$$\mathcal{W}_o \omega_\times \overline{\omega}_\times = \mathcal{H}_o \overline{\mathcal{H}_o} \tag{46}$$

And finally we have to eliminate the physical magnetic charges $\mu_0^{(ph)}, \mu_s^{(ph)}$ of the sources. In accordance with (40)-(43), bearing in mind that $\delta = 0$ in agreement with (45), we have to add the following two conditions:

$$\mu_0 + \mathrm{Im}\left[(q_\times + i\mu_\times)\frac{(a_\times + i\sigma_s - im_\times + b_\times)}{2(a_\times + i\sigma_s)} - \frac{\mathcal{L}_0 \mathcal{H}_0}{2\mathcal{W}_o(a_\times + i\sigma_s)}\right] = 0 \tag{47}$$

$$\mu_\times - \mathrm{Im}\left[(q_\times + i\mu_\times)\frac{(a_\times + i\sigma_s - im_\times + b_\times)}{2(a_\times + i\sigma_s)} - \frac{\mathcal{L}_0 \mathcal{H}_0}{2\mathcal{W}_o(a_\times + i\sigma_s)}\right] = 0 \tag{48}$$

9 Evidence of the existing of the physical equilibrium

The easiest way to prove that equilibrium equations have physically acceptable solutions is to consider the limit of large distance between the sources. Assuming that l is much larger than all other parameters we can try to find an expansion for the solution of equations (44)-(48) with respect to the quantity $1/l$. It turns out that solution in the form of such expansion indeed exists. To show this we assume that six parameters $m_0, a_0, q_0, m_s, a_s, q_s$ are independent of l while the four constants b_0, μ_0, b_s, μ_s follows from the equilibrium equations (44)-(48) as functions of l and of the aforementioned six parameters $m_0, a_0, q_0, m_s, a_s, q_s$. Then, in the limit of large l, these functions can be expanded with respect to $1/l$. The calculations show that these expansions are:

$$b_0 = \frac{m_0\left(a_s + i\sqrt{m_s^2 - a_s^2 - q_s^2}\right)}{l} + O\left(\frac{1}{l^2}\right), \tag{49}$$

$$\mu_0 = -\frac{q_0\left(a_s + i\sqrt{m_s^2 - a_s^2 - q_s^2}\right)}{l} + O\left(\frac{1}{l^2}\right), \tag{50}$$

$$b_s = -\frac{m_s a_0}{l} + O\left(\frac{1}{l^2}\right), \tag{51}$$

$$\mu_s = \frac{q_s a_0}{l} + O\left(\frac{1}{l^2}\right). \tag{52}$$

Here, in accordance with the initial inequalities (5), $m_s^2 - a_s^2 - q_s^2 < 0$ then all parameters (49)-(52) are real as it should be. Using this result it is easy to obtain from (18) and (20) the total mass and total electric charge of the configuration:

$$M = m_0 + m_s + \frac{m_s\left(\sqrt{m_0^2 - a_0^2 - q_0^2} - \Lambda\right)}{l} + O\left(\frac{1}{l^2}\right), \tag{53}$$

$$Q_e = q_0 + q_s + \frac{q_s\left(\sqrt{m_0^2 - a_0^2 - q_0^2} - \Lambda\right)}{l} + O\left(\frac{1}{l^2}\right). \tag{54}$$

where

$$\Lambda = -\frac{\sqrt{m_0^2 - a_0^2 - q_0^2}\,(m_0 q_s - m_s q_0)^2}{(m_0^2 - q_0^2)\left(m_s^2 - a_s^2 - q_s^2 - i\sqrt{m_s^2 - a_s^2 - q_s^2}\right)}\ . \tag{55}$$

We remind that, again in accordance with the inequalities (5), the quantity $m_0^2 - a_0^2 - q_0^2$ is positive.

There is one additional important result which is coming from the solution of the system (44)-(48): the basic (independent of l) parameters m, q_0, m_s, q_s must satisfy the relation

$$m_0 m_s = q_0 q_s\ . \tag{56}$$

This equation follows from (45), that is from the absence of the conic syngularities at the points of symmetry axis. Then the total amount of the equilibrium constraints on the initial 11 parameters m_0, a_0, b_0, q_0, μ_0, m_s, a_s, b_s, q_s, μ_s, l are (49)-(52) and (56), that is five conditions from five equations (44)-(48) as it should be. Consequently the solution describing the physical equilibrium state of two charged rotating sources contains six arbitrary constants. This is by two constants more then the corresponding static solution we found in [3] which is natural since in case of rotating sources the two rotation parameters a_0 and a_s appeared.

It is worth to mention that in derivation of the expressions (49)-(55) we used the condition (56) which lead to the essential simplifications of these formulas. Another remark we would like to add is the fact that formulas (49)-(54) we obtained up to the order $1/l^3$. These additional details showed no surprises and there is no of big interest to exhibit them here.

The last remark relates to the total angular momentum of the configuration. To find it it is necessary to expand the Ernst potential $\mathcal{E}$ at spatial infinity up to the order $1/r^2$ choosing an appropriate spherical coordinates r, θ. Then the angular momentum J reveal itself in the summand $2iJr^{-2}\cos\theta$ in $\mathcal{E}$. We calculated it but the expression is rather long and we will not show it here. The only interesting fact is that in the limit of large l, discussed above, the total angular momentum is

$$J = m_0 a_0 + m_s a_s + O\left(\frac{1}{l}\right). \tag{57}$$

This result together with relations $M = m_0 + m_s$ and $Q_e = q_0 + q_s$ following from (53)-(54) in the first non-vanishing approximation and relation $m_0 m_s = q_0 q_s$ show that at the infinite separation between the sources they are just usual electrically charged rotating Kerr-Newman objects without any NUT annoyance and without magnetic charges.

Of course the analysis presented in this section can be consider as a proof of existence of the equilibrium for the distances l from the region $l_{cr} < l < \infty$ but the critical value l_{cr} can be extracted only from the exact solution of the equilibrium equations (44)-(48) (it is probable that in general l_{cr} is simply zero under an appropriate definition of the distance). This task we postpone for a future work.

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